Assignment 5.

This homework is due *Friday* Feb 21.

There are total 30 points in this assignment. 27 points is considered 100%. If you go over 27 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

(1) (a) [2pt] (4.2.1a) Find the remainders when 2^{50} and 41^{65} are divided by 7.

(b) [3pt] (∼4.2.1b) What is the remainder when the following sum is divided by 4?

$$1^5 + 2^5 + 3^5 + \ldots + 2013^5 + 2014^5$$

(c) [3pt] (part of 4.1.5) Prove that $53^{103} + 103^{53}$ is divisible by 39.
(*Hint:* $39 = 13 \cdot 3$.)

(d) [3pt] (part of 4.1.5) Prove that $111^{333} + 333^{111}$ is divisible by 7.

(2) (a) [3pt] Prove that if p is a prime and $1 \le k \le p-1$, then $p \mid \binom{p}{k}$.

(b) [2pt] Prove that $(a+b)^p \equiv a^p + b^p \pmod{p}$ for any integers a, b.

(c) [2pt] Prove that $(a_1 + a_2 + \ldots + a_n)^p = a_1^p + a_2^p + \ldots + a_n^p \pmod{p}$ for any integers $a_1, \ldots, a_n$.
(*Hint:* $a_1 + a_2 + \ldots + a_n = (a_1 + \ldots + a_{n-1}) + a_n$. Use induction.)

(d) [2pt] Prove that for any integer a one has $a^p \equiv a \pmod{p}$.
(*Hint:* If $a > 0$, then $a = 1 + 1 + \ldots + 1$.)

COMMENT. Congratulations, you just proved Fermat's Little Theorem. I will give a different proof in class at some point.

(3) [2pt] (4.3.16) Show that 2^n divides an integer N if and only if 2^n divides the number made up of the last n digits of N. (*Hint:* $2^k 5^k = 10^k \equiv 0 \pmod{2^n}$ for $k \ge n$.)

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(4) The International Standard Book Number (ISBN-10) used in many libraries (and found on most books printed by an actual publisher) consists of nine digits $a_1, a_2, \ldots, a_9$, followed by a tenth check number a_{10} that satisfies

$$a_{10} \equiv \sum_{k=1}^{9} k a_k \pmod{11}. \tag{\#}$$

(So, a_{10} is a digit or a 10.)

(a) [2pt] (4.3.27) Determine whether each of ISBNs below is valid:
0-07-232569-0,
91-76-43-497-5,
1-56947-303-10.

(b) [3pt] (4.3.28) When printing ISBN $a_1 a_2 \ldots a_9$, two unequal digits were transposed. Show that the check digit detected that an error happened. (That is, the check digit no longer satisfies formula (#).)

(c) [3pt] Prove that the following condition on a_{10} is equivalent to the one given by formula (#) above:

$$10a_1 + 9a_2 + \ldots + 2a_9 + 1a_{10} \equiv 0 \pmod{11}.$$